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Linear algebra

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Problem Set # 7

Due to next monday in class

Exercise 1:

1. (Dimension theorem) If W is a subspace of a K-vector space V, then
$$dim_K(W) + dim_K(W^\circ) = dim_K(V)$$
2. If $T : V \to W$ is a linear operator of K vector space, prove that $K(T^t) =$ the annihilator $(R(T))^\circ$ of $range(T)$.
3. If $T : V \to V$ is linear and W a subspace of V. Prove that W is T-invariant if and only if W° is invariant under the adjoint operator T^t
4. If $T : V \to W$ is linear. Prove that $rank(T^t) = rank(T)$.
5. If $A \in M_{n\times n}(K)$. We define $L_A : K^n \to K^n$ via $L_A(v) = A \cdot v$, for $v \in K^n$. The rank of any linear operator T is the dimension of the range $R(T)$. Prove that
$$rank(L_A) = rank(L_{A^t}) = colrank(A) = colrank(A^t) = rowrank(A) = colrank(A^t)$$
(Recall that
$$colrank(A) = dim(K - Span\{columns\ col_i(A)\ in\ A\})$$
and
$$rowrank(A) = dim(K - Span\{rows\ col_i(A)\ in\ A\})).$$

Exercise 2:

1. Write
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 1 & 3 \end{pmatrix}$$
as a product of disjoint commuting cycles.
2. Evaluate the net action of the following products of cycles
   (a) $(1, 2)(1, 3)$ in S_3;
   (b) $(1, 2, 3)^2 (= (1, 2, 3)(1, 2, 3))$ in S_5.

3. Determine the inverses σ^{-1} of the following elements in S_5
   (a) Any 2-cycle (i_1, i_2) with $i_1 \neq i_2$;
   (b) Any k-cycle $(i_1, ..., i_k)$.
4. Evaluate the products in S_n as products of disjoint cycles
   (a) $(1,5)(1,4)(1,3)(1,2)$;
   (b) $(1,k)(1,2,...,k-1)$.

Exercise 3:

1. Prove that $(i_1, ..., i_k) = (i_1, i_k)(i_1, i_{k-1})...(i_1, i_2)$ and deduce that any permutation can be written as product of 2-cycles using a theorem mentioned in class; (Note that this decomposition is far from being unique.)
2. Consider the polynomial in n-unknowns $\phi \in K[x_1, ..., x_n]$ given by $\phi(x_1, ..., x_n) = \prod_{i<j}(x_i - x_j)$. The group S_n acts on $K[x_1, ..., x_n]$ via permutation of the variables

$$\sigma \cdot f(x_1, ..., x_n) = f(x_{\sigma(1)}, ..., x_{\sigma(n)})$$

   (a) Check that this is a "covariant group action" in the sense that $(\sigma\tau) \cdot f = \sigma \cdot (\tau \cdot f)$ for all $\sigma, \tau \in S_n$ and $e \cdot f = f$ for the identity element e and all $f \in K[x_1, ..., x_n]$.
   (b) Prove that $\sigma \cdot \phi = (-1) \cdot \phi$ for any 2-cycle $\sigma = (i, j)$.
   (c) Deduce that $\sigma \cdot \phi = (-1)^r \phi$ if σ is a product $\tau_1, ..., \tau_r$ of r two-cycles.
   (d) Let $\sigma \in S_n$ and $\sigma = \tau_1, ..., \tau_r$ and $\sigma = \tau'_1, ..., \tau'_s$ two different decomposition in 2-cycles. Deduce for the previous question that $(-1)^r = (-1)^s$.

   We denote this number is $sgn(\sigma) = (-1)^r$ which is the "parity" of a permutation where r is the number of 2-cycles in the factorization $\sigma = \tau_1 ... \tau_r$ (NB: we have just proved that it does not depend on the decomposition in 2-cycle, so the definition makes sense).

   (e) Prove that $sgn(e) = 1$, $sgn(\sigma\tau) = sgn(\sigma)sgn(\tau)$, and $sgn(\sigma^{-1}) = sgn(\sigma)$, for any $\sigma, \tau \in S_n$.